RAABE'S TEST

(B.Sc.-II, Paper-III)

Group- B

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1.

Raabe's Test

Theorem (Raabe's test) :→ If Σa_n is a series of positive terms such that

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = l$$

then (i) Σa_n is convergent if $l > 1$

(ii) Σa_n is divergent if $l < 1$.

Proof:→

Case (i) If $l > 1$. then

Let $1 + k = l$

Let $\epsilon = \frac{k}{2}$

$$\because \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = l,$$

Therefor,

$$l - \epsilon < n\left(\frac{a_n}{a_{n+1}} - 1\right) < l + \epsilon, \ \forall n \geq m.$$

$$\Rightarrow 1 + k - \frac{k}{2} < n\left(\frac{a_n}{a_{n+1}} - 1\right), \ \forall n \geq m.$$

$$\Rightarrow \frac{k}{2} < \frac{na_n - na_{n+1}}{a_{n+1}} - 1, \ \forall n \geq m.$$

$$\therefore \frac{k}{2} \cdot a_{n+1} < na_n - (n+1)a_{n+1}, \ \forall n \geq m.$$

We put $n = m+1, m+2, \ldots, r-1$; $(r \geq m+2)$

$\therefore \frac{K}{2}(a_{m+2}) < (m+1)a_{m+1} - (m+2)a_{m+2}$

$\frac{K}{2}(a_{m+3}) < (m+2)a_{m+2} - (m+3)a_{m+3}$.

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$\frac{K}{2}a_r < (r-1)a_{r-1} - r a_r$

Adding all these inequalities

$\frac{K}{2}(a_{m+2} + a_{m+3} + \cdots + a_r) < (m+1)a_{m+1} - r a_r$

$\Rightarrow \frac{K}{2}(a_{m+2} + a_{m+3} + \cdots + a_r) < (m+1)a_{m+1}, \quad \forall r \geq m+2$

$\Rightarrow a_{m+2} + a_{m+3} + \cdots + a_r < \frac{2}{K}\cdot(m+1)a_{m+1}, \quad \forall r \geq m+2.$

Therefor,

$a_1 + a_2 + a_3 + \cdots + a_r < a_1 + a_2 + \cdots + a_m + \frac{2}{K}\cdot(m+1)a_{m+1}, \forall r \geq m+2.$

$\therefore S_r < M$ (constant), $\forall r \geq m+2$.

Also since (S_r) is monotonic increasing sequence ($\because a_n > 0$)

$\therefore (S_r)$ is convergent.

$\Rightarrow \sum a_r$ is convergent.

Case-(iii) :→ If $l < 1$, then

$$\epsilon = 1 - l > 0$$

∴ By definition of the limit.

$$l - \epsilon < n\left(\frac{a_n}{a_{n+1}} - 1\right) < l + \epsilon, \quad \forall\, n \geq m.$$

$$\Rightarrow \quad n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1, \quad \forall\, n \geq m.$$

$$\Rightarrow \quad n \cdot \left(\frac{a_n - a_{n+1}}{a_{n+1}}\right) < 1, \quad \forall\, n \geq m.$$

$$\Rightarrow \quad na_n - na_{n+1} < a_{n+1} \quad (\because a_{n+1} > 0), \forall\, n \geq m.$$

$$\therefore \quad na_n < (n+1)\, a_{n+1}; \quad \forall\, n \geq m.$$

putting $n = m+1, m+2, \text{--------}, r-1$ (where $r \geq m+2$) in above inequality, we get

$$(m+1)\, a_{m+1} < (m+2)\, a_{m+2} < (m+3)\, a_{m+3} < \cdots$$

$$\cdots\cdots < (r-1)\, a_{r-1} < r a_r, \quad \forall\, r \geq m+2.$$

$$\Rightarrow (m+1)\, a_{m+1} < r a_r, \text{ for all } r \geq m+2.$$

$$\Rightarrow \quad a_r > \frac{D}{r}, \text{ where } D = (m+1)\, a_{m+1} = \text{constant}$$

$$\forall\, r \geq m+2.$$

∵ The series $\sum \frac{1}{r}$ is <u>divergent</u>.

∴ By comparision test.

$\sum a_r$ is divergent.

proved.

Example ① :→ Test for convergence the series

$$\frac{1}{2}x + \frac{1\cdot 2}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots\cdots$$

for all positive x.

Solution :→

$\because x > 0$, the series is positive term series.

Here, $a_n = \dfrac{1\cdot 3\cdot 5\cdots\cdots(2n-1)}{2\cdot 4\cdot 6\cdots\cdots(2n)}\cdot x^n$,

and $a_{n+1} = \dfrac{1\cdot 3\cdot 5\cdots\cdots(2n-1)\cdot(2n+1)}{2\cdot 4\cdot 6\cdots\cdots(2n)\ (2n+2)}\cdot x^{n+1}$,

$$\therefore \lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{(2n+2)}{(2n+1)}\cdot\frac{1}{x}$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{2n}\right)}\cdot\frac{1}{x}$$

$$\Rightarrow \lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \frac{1}{x}$$

$\therefore$ By D'Alembert's ~~test~~ ratio test.

The series is convergent if $x < 1$. $(0 < x < 1)$

divergent if $x > 1$.

When $x = 1$, then $\lim\limits_{n\to\infty}\dfrac{a_n}{a_{n+1}} = 1$, and the D'Alembert ratio test fails.

$$\therefore \frac{a_n}{a_{n+1}} - 1 = \frac{2n+2}{2n+1} - 1 = \frac{1}{2n+1}.$$

$$\therefore n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{n}{2n+1}$$

$$\therefore \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}}$$

$$= \frac{1}{2} < 1.$$

$\therefore$ By Raabe's test.

The series is divergent at $x=1$.

Example② Test for convergence the series.

$$1 + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} + \cdots\cdots$$

Where $\alpha, \beta \geq 0$.

Solution:→

Here $a_n = \dfrac{(1+\alpha)(2+\alpha)\cdots\cdots(n-1+\alpha)}{(1+\beta)(2+\beta)\cdots\cdots(n-1+\beta)}$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+\alpha}{n+\beta} = \lim_{n\to\infty} \frac{1+\frac{\alpha}{n}}{1+\frac{\beta}{n}} = 1.$$

$\therefore$ D'Alembert's ratio test fails.

$$\because \quad n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{n+\beta}{n+\alpha} - 1\right)$$

$$= n\left(\frac{n+\beta-n-\alpha}{n+\alpha}\right)$$

$$\therefore \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{n(\beta-\alpha)}{n+\alpha}$$

$$= \lim_{n\to\infty} \frac{\beta-\alpha}{1+\frac{\alpha}{n}} = \beta-\alpha.$$

Thus by Raabe's test, the series is convergent if $\beta-\alpha > 1$ or $\beta > 1+\alpha$, and divergent if $\beta < \alpha+1$.

For $\beta = \alpha + 1$, the series becomes $\sum \frac{1+\alpha}{n+\alpha}$

Taking $b_n = \frac{1}{n}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n(1+\alpha)}{n+\alpha}$

$$= \lim_{n\to\infty} \frac{1+\alpha}{1+\frac{\alpha}{n}}$$

$$= 1+\alpha \quad (\text{finite})$$

$\therefore$ By comparision test

$\because \sum \frac{1}{n}$ is divergent

$\therefore \sum \frac{1+\alpha}{n+\alpha}$ is divergent.

Thank you